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A New Quadrature Formula Associated with the Ultraspherical Polynomials

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A new Birkhoff-type quadrature formula associated with the extended ultraspherical nodes is obtained and applied to evaluate the second kind of ultraspherical polynomials at the endpoints of the interval of orthogonality.

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For a given $\alpha > -1$ let $\{P_n^{(\alpha)}\}_{n=0}^{\infty}$ denote the system of ultraspherical polynomials orthogonal in $[-1, 1]$ with respect to the weight function $(1-x^2)^\alpha$ and normalized by $P_n^{(\alpha)}(1) = \binom{n+\alpha}{n}$. In [8] the second of us obtained some new quadrature formulas based on the zeros

$$1 = x_{1n} > x_{2n} > \cdots > x_{nn} = -1$$

of

$$\pi_n(x) = (1-x^2)\, P_{n-1}^{(0)\prime}(x).$$

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In particular, it was proved in [8, p. 799] that the quadrature formula

$$\begin{aligned}\int_{-1}^{1} f(x)\,dx &= \frac{3}{n(2n-1)}\,[f(1)+f(-1)] \\ &\quad + \frac{2(2n-3)}{n(n-2)(2n-1)} \sum_{k=2}^{n-1} \frac{f(x_{kn})}{(P_{n-1}^{(0)}(x_{kn}))^2} \\ &\quad + \frac{1}{n(n-1)(n-2)(2n-1)} \sum_{k=2}^{n-1} \frac{(1-x_{kn}^2)f''(x_{kn})}{(P_{n-1}^{(0)}(x_{kn}))^2} \end{aligned} \tag{1.1}$$

is exact for all polynomials f of degree at most $2n-1$, and thereby problems 36–39 in Turán [7] were solved.

The object of this paper is to find quadrature formulas analogous to (1.1). Let us denote by

$$1 = t_{0n}^{(\alpha)} > t_{1n}^{(\alpha)} > \cdots > t_{nn}^{(\alpha)} > t_{n+1,n}^{(\alpha)} = -1$$

the zeros of $(1-x^2)\,P_n^{(\alpha)}(x)$, and let λ_{kn}, $k = 1, 1,..., n$, be the Cotes numbers of the Gauss–Jacobi quadrature process associated with the ultraspherical weight function, that is,

$$\lambda_{kn}^{(\alpha)} = \int_{-1}^{1} \frac{P_n^{(\alpha)}(x)}{(P_n^{(\alpha)\prime}(t_{kn}))(x-t_{kn})}\,(1-x^2)^\alpha\,dx. \tag{1.2}$$

Our main result is the following

THEOREM 1. *For every $\alpha > -1$ and $n = 1, 2,...$ the quadrature formula*

$$\begin{aligned}&\int_{-1}^{1} f(x)(1-x^2)^\alpha\,dx \\ &\quad = \frac{-2^{2\alpha}\Gamma(\alpha+1)\,\Gamma(\alpha+2)\,\Gamma(n+1)}{(n-\alpha-1)(2n+2\alpha+1)\,\Gamma(n+2\alpha+1)}\,[f(1)+f(-1)] \\ &\quad + \frac{n(2n-1)}{(n-\alpha-1)(2n+2\alpha+1)} \sum_{k=1}^{n} \lambda_{kn}^{(\alpha)} f(t_{kn}^{(\alpha)}) \\ &\quad - \frac{2\alpha(1+\alpha)}{(n-\alpha-1)(2n+2\alpha+1)} \sum_{k=1}^{n} \frac{\lambda_{kn}}{1-(t_{kn}^{(\alpha)})^2} f(t_{kn}^{(\alpha)}) \\ &\quad + \frac{1}{2(n-\alpha-1)(2n+2\alpha+1)} \sum_{k=1}^{n} (1-t_{kn}^{(\alpha)})^2)\,\lambda_{kn}^{(\alpha)} f''(t_{kn}^{(\alpha)}) \end{aligned} \tag{1.3}$$

is exact for all polynomials f of degree at most $2n+1$.

The proof of Theorem 1 is essentially different from the one given for (1.1) in [8], and the main idea of the proof is to apply the Gauss–Jacobi quadrature formula [6, p. 47] and Theorem 3 which is stated below.

Theorem 1 can be applied to evaluate the second kind of ultraspherical polynomials at ± 1. Let us define the second kind of ultraspherical polynomials $Q_n^{(\alpha)}$ associated with $P_n^{(\alpha)}$ by the formula

$$Q_n^{(\alpha)}(x)=\int_{-1}^{1}\frac{P_n^{(\alpha)}(x)-P_n^{(\alpha)}(t)}{x-t}(1-t^2)^\alpha\,dt. \tag{1.4}$$

It is well known that the second kind (or numerator) of polynomials $Q_n^{(\alpha)}$ satisfies the same recurrence formula which generates $P_n^{(\alpha)}$ except that the initial data are $Q_0^{(\alpha)}=0$ and $Q_1^{(\alpha)}=\sqrt{\pi}\,\Gamma(\alpha+2)/\Gamma(\alpha+\frac{3}{2})$ (see, e.g., [2–4, 6]). By the Gauss–Jacobi quadrature formula [6, p. 47] we can express (1.4) as

$$Q_n^{(\alpha)}(x)=P_n^{(\alpha)}(x)\sum_{k=1}^{n}\frac{\lambda_{kn}^{(\alpha)}}{x-t_{kn}^{(\alpha)}},$$

and applying (1.3) with $f(t)\equiv 1+t$ we obtain after some elementary computations the following

THEOREM 2. *Let $a>-1$, $\alpha=0$. Then, for $n=1,2,...,$*

$$(\pm 1)^n\,Q_n^{(\alpha)}(\pm 1)=\frac{\sqrt{\pi}}{\alpha\Gamma(\alpha+\frac{1}{2})}\,\frac{\Gamma(n+\alpha+1)}{n!}-\frac{2^{2\alpha}\Gamma(\alpha+1)}{\alpha}\,\frac{\Gamma(n+\alpha+1}{\Gamma(n+2\alpha+1)}.$$

Remark. Theorem 2 may also be obtained from generating functions for the second kind of ultraspherical polynomials [1, 2]. This was pointed out to us by both R. Askey and M. Ismail in independent private communications. R. Askey has also informed us that a generalization of Theorem 2 was proved in [9].

Some special cases of Theorem 1 deserve individual attention. Namely, when $\alpha=-\frac{1}{2}$ then (1.3) becomes

$$\begin{aligned}\int_{-1}^{1} f(x)(1-x^2)^{-1/2}\,dx &= -\frac{\pi}{4(2n-1)}\,[f(1)+f(-1)]\\ &\quad+\frac{\pi}{n}\sum_{k=1}^{n}\left(1+\frac{1}{2n(2n-1)(1-(t_{kn}^{(-1/2)})^2)}\right)f(t_{kn}^{(-1/2)})\\ &\quad+\frac{\pi}{2n^2(2n-1)}\sum_{k=1}^{n}(1-(t_{kn}^{(-1/2)})^2)\,f''(t_{kn}^{(-1/2)})\end{aligned} \tag{1.5}$$

where $t_{kn}^{(-1/2)} = \cos(k - \frac{1}{2})\,(\pi/n)$ are the zeros of the Chebyshev polynomials, whereas for $\alpha = 0$ formula (1.3) takes the form

$$\int_{-1}^{1} f(x)\,dx = -\frac{1}{(n-1)(2n+1)}\,[f(1)+f(-1)] + \frac{2n(2n-1)}{(n-1)(2n+1)} \sum_{k=1}^{n} \frac{1}{(1-t_{kn}^2)(P_n'(t_{kn}))^2} f(t_{kn}) + \frac{1}{(n-1)(2n+1)} \sum_{k=1}^{n} \frac{1}{(P_n'(t_{kn}))^2} f''(t_{kn}) \quad (1.6)$$

where t_{kn} are the zeros of the Legendre polynomials $P_n \equiv P_n^{(0)}$.

The proof of Theorem 1 is based on the following

THEOREM 3. *Let $\alpha > -1$ and $n = 1, 2, \ldots$. Then the quadrature formula*

$$\int_{-1}^{1} f(x)(1-x^2)^\alpha\,dx = 2^{2\alpha} \frac{(\Gamma(\alpha+1))^2\,\Gamma(n+1)}{(2n+2\alpha+1)\,\Gamma(n+2\alpha+1)}\,[f(1)+f(-1)] + \frac{2}{(2n+2\alpha+1)} \sum_{k=1}^{n} \left(n + \frac{\alpha}{1-(t_{kn}^{(\alpha)})^2}\right) \lambda_{kn}^{(\alpha)} f(t_{kn}^{(\alpha)}) - \frac{1}{2n+2\alpha+1} \sum_{k=1}^{n} t_{kn}^{(\alpha)}\,\lambda_{kn}^{(\alpha)} f'(t_{kn}^{(\alpha)}) \quad (1.7)$$

holds for every polynomial f of degree at most $2n+1$.

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In this section we will prove Theorems 1 and 3. For the sake of brevity we will omit some unessential indices in the formulas for example, we will write λ_k and t_k instead of $\lambda_{kn}^{(\alpha)}$ and $t_{kn}^{(\alpha)}$, respectively.

Proof of Theorem 3. Fix $\alpha > -1$ and $n = 1, 2, \ldots$, and let f be an arbitrary polynomial of degree at most $2n+1$. Then by the Hermite interpolation formula (see, e.g., [5])

$$f(x) = \sum_{k=0}^{n+1} f(t_k)\,r_k(x) + \sum_{k=1}^{n} f'(t_k)\,q_k(x) \quad (2.1)$$

where

$$r_0(x)=\binom{n+\alpha}{n}^{-2} 2^{-1}(1+x)(P_n^{(\alpha)}(x))^2, \tag{2.2}$$

$$r_{n+1}(x)=\binom{n+\alpha}{n}^{-2} 2^{-1}(1-x)(P_n^{(\alpha)}(x))^2, \tag{2.3}$$

and for $k=1, 1,..., n$

$$r_k(x)=\frac{1+c_k(x-t_k)}{1-t_k^2}(1-x^2)\, l_k^2(x), \tag{2.4}$$

$$c_k=-2\alpha\frac{t_k}{1-t_k^2}, \tag{2.5}$$

$$q_k(x)=\frac{(x-t_k)}{1-t_k^2}(1-x^2)\, l_k^2(x), \tag{2.6}$$

and

$$l_k(x)=\frac{P_n^{(\alpha)}(x)}{P_n^{(\alpha)\prime}(t_k)(x-t_k)}. \tag{2.7}$$

We can apply (2.2), (2.3), and formula (4.33) in [6, p. 68] to obtain

$$\int_{-1}^{1} r_k(x)(1-x^2)^\alpha\, dx$$

$$=2^{2\alpha}\frac{(\Gamma(\alpha+1))^2\,\Gamma(n+1)}{(2n+2\alpha+1)\,\Gamma(n+2\alpha+1)},\qquad k=0, n+1. \tag{2.8}$$

It follows from (1.2), (2.7), and orthogonality relations that

$$\int_{-1}^{1}(x-t_k)\, l_k^2(x)(1-x^2)^\alpha\, dx=0 \tag{2.9}$$

and

$$\int_{-1}^{1} l_k(x)(1-x^2)^\alpha\, dx=\int_{-1}^{1} l_k^2(x)(1-x^2)^\alpha\, dx=\lambda_k \tag{2.10}$$

for $k=1, 2,..., n$. Applying (2.6), (2.7), and (2.9) we obtain

$$\int_{-1}^{1} q_k(x)(1-x^2)^\alpha\, dx$$

$$=(1-t_k^2)^{-1}\int_{-1}^{1}(x-t_k)(t_k^2-x^2)\, l_k^2(x)(1-x^2)^\alpha\, dx$$

$$=-(1-t_k^2)^{-1}(P_n^{(\alpha)\prime}(t_k))^{-2}\int_{-1}^{1}(x+t_k)(P_n^{(\alpha)}(x))^2(1-x^2)^\alpha\, dx$$

and thus by (4.3.3) and (15.3.1) in [6, pp. 68 and 352, resp.]

$$\int_{-1}^{1} q_k(x)(1-x^2)^\alpha\, dx = -\frac{1}{2n+2\alpha+1}\, t_k \lambda_k \tag{2.11}$$

for $k=1, 2,..., n$. By (2.4), (2.5), (2.6), and (2.11) we have

$$\begin{aligned}\int_{-1}^{1} & r_k(x)(1-x^2)^\alpha\, dx \\ &= (1-t_k^2)^{-1} \int_{-1}^{1} l_k^2(x)(1-1x^2)^{\alpha+1}\, dx \\ &\quad + \frac{2\alpha t_k^2 \lambda_k}{(2n+2\alpha+1)(1-t_k^2)},\end{aligned} \tag{2.12}$$

$k=1, 2,..., n$. Using (2.7), (2.9), and (2.10) we get

$$\begin{aligned}\int_{-1}^{1} & l_k^2(x)(1-x^2)^{\alpha+1}\, dx \\ &= \int_{-1}^{1} l_k^2(x)(1-t_k^2+t_k^2-x^2)(1-x^2)^\alpha\, dx \\ &= (1-t_k^2)\,\lambda_k - \int_{-1}^{1} (x-t_k)^2\, l_k^2(x)(1-x^2)^\alpha\, dx \\ &= (1-t_k^2)\,\lambda_k - (P_n^{(\alpha)\prime}(t_k))^{-2} \int_{-1}^{1} (P_n^{(\alpha)}(x))^2(1-x^2)^\alpha\, dx\end{aligned}$$

and applying again (4.3.3) and (15.3.1) in [6] we obtain from (2.12)

$$\begin{aligned}\int_{-1}^{1} & r_k(x)(1-x^2)^\alpha\, dx \\ &= \lambda_k - \frac{\lambda_k}{2n+2\alpha+1} + \frac{2\alpha t_k^2 \lambda_k}{(2n+2\alpha+1)(1-t_k^2)} \\ &= \frac{2n}{2n+2\alpha+1}\,\lambda_k + \frac{2\alpha}{2n+2\alpha+1}\,\frac{\lambda_k}{1-t_k^2},\end{aligned} \tag{2.13}$$

$k=1, 2,..., n$. Now Theorem 3 follows immediately from (2.1), (2.8), (2.11), and (2.13).

Proof of Theorem 1. Let us fix $\alpha > -1$ and $n = 1, 2, \ldots$. If f is identically 1 then by the Gauss–Jacobi quadrature formula (1.3) becomes

$$\int_{-1}^{1} (1-x^2)^\alpha \, dx = -2^{2\alpha+1} \frac{\Gamma(\alpha+1)\,\Gamma(\alpha+2)\,\Gamma(n+1)}{(n-\alpha-1)(2n+2\alpha+1)\,\Gamma(n+2\alpha+1)} + \frac{n(2n-1)}{(n-\alpha-1)(2n+2\alpha+1)} \int_{-1}^{1} (1-x^2)^\alpha \, dx - \frac{2\alpha(1+\alpha)}{(n-\alpha-1)(2n+2\alpha+1)} \sum_{k=1}^{n} \frac{\lambda_k}{1-t_k^2}, \tag{2.14}$$

whereas by (1.7) and the Gauss–Jacobi quadrature formula

$$\int_{-1}^{1} (1-x^2)^\alpha \, dx = \frac{2^{2\alpha+1}(\Gamma(\alpha+1))^2\,\Gamma(n+1)}{(2n+2\alpha+1)\,\Gamma(n+2\alpha+1)} + \frac{2n}{2n+2\alpha+1} \int_{-1}^{1} (1-x^2)^\alpha \, dx + \frac{2\alpha}{2n+2\alpha+1} \sum_{k=1}^{n} \frac{\lambda_k}{1-t_k^2}. \tag{2.15}$$

It is a matter of a simple exercise involving gamma and beta functions to show that (2.14) and (2.15) are equivalent, and therefore (1.3) holds for $f \equiv 1$. If f is any odd polynomial then by symmetry all sums and integrals in (1.3) equal 0 and thus (1.3) holds again. Now let f be an even polynomial of degree at most $2n-1$ vanishing at ± 1. Then by the Gauss–Jacobi quadrature formula (1.3) is equivalent to

$$\int_{-1}^{1} f(x)(1-x^2)^\alpha \, dx = \frac{n(2n-1)}{(n-\alpha-1)(2n+2\alpha+1)} \int_{-1}^{1} f(x)(1-x^2)^\alpha \, dx - \frac{2\alpha(1+\alpha)}{(n-\alpha-1)(2n+2\alpha+1)} \int_{-1}^{1} f(x)(1-x^2)^{\alpha-1} \, dx + \frac{1}{2(n-\alpha-1)(2n+2\alpha+1)} \int_{-1}^{1} f''(x)(1-x^2)^{\alpha+1} \, dx,$$

that is,

$$(2\alpha+1) \int_{-1}^{1} f(x)(1-x^2)^\alpha \, dx = 2\alpha \int_{-1}^{1} f(x)(1-x^2)^{\alpha-1} \, dx - \frac{1}{2(1+\alpha)} \int_{-1}^{1} f''(x)(1-x^2)^{\alpha+1} \, dx. \tag{2.16}$$

Twofold integration by parts of the second integral on the right side of (2.16) immediately demonstrates the validity of (2.16), and hence (1.3) holds again. If $f(x) \equiv (P_n^{(\alpha)}(x))^2$ then (1.3) becomes

$$\int_{-1}^{1} (P_n^{(\alpha)}(x))^2(1-x^2)^\alpha \, dx$$

$$= -2^{2\alpha+1} \frac{\Gamma(\alpha+1)\,\Gamma(\alpha+2)\,\Gamma(n+1)}{(n-\alpha-1)(2n+2\alpha+1)\,\Gamma(n+2\alpha+1)} (P_n^{(\alpha)}(1))^2$$

$$+ \frac{1}{(n-\alpha-1)(2n+2\alpha+1)} \sum_{k=1}^{n} (1-t_k^2)\,\lambda_k (P_n^{(\alpha)\prime}(t_k))^2. \qquad (2.17)$$

Applying now (4.3.3) and (15.3.1) in [6, pp. 68 and 352] to (2.17), formula (1.3) follows again. Finally, if f is an arbitrary polynomial of degree at most $2n+1$ then we can decompose it into $f = f_1 + f_2 + f_3 + f_4$ where f_1 is constant, f_2 is odd, f_3 is an even polynomial of degree at most $2n-1$ vanishing at ± 1, and f_4 is proportional to $(P_n^{(\alpha)})^2$. Since we have proved (1.3) for each f_i $(i = 1, 2, 3, 4)$ in this decomposition, the validity of (1.3) follows for every polynomial f of degree at most $2n+1$.

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